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Generalized Zernike or disc polynomials

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Abstract

We investigate generalized Zernike or disc polynomials $P^{\alpha}_{m,n}(z,z^*)$ which are orthogonal 2D polynomials in the unit disc $0 \leqslant \sqrt{zz^*} < 1$ with weights $(1-zz^*)^{\alpha}$ in complex coordinates $z \equiv x + \mathrm{i}y$, $z^* \equiv x - \mathrm{i}y$, where $\alpha > -1$ is a free parameter. These polynomials can be expressed by Jacobi polynomials of transformed arguments in connection with a simple angle dependence. A limiting procedure $\alpha \to \infty$ leads to Laguerre 2D polynomials $L_{m,n}(z,z^*)$. Furthermore, we introduce the corresponding orthonormalized disc functions. The disc polynomials and disc functions obey two differential equations, a first-order and a second-order one with a certain degree of freedom, and the operators of lowering and raising of the indices are found. These operators can be closed to a Lie algebra $\mathrm{su}(1,1) \oplus \mathrm{su}(1,1)$. New generating functions are derived from an operational representation which is alternative to the Rodrigues-type representation. The one-dimensional analogue of the disc polynomials which are orthogonal polynomials in the interval $0 \leqslant r \leqslant 1$ with weight factors $(1-r^2)^{\alpha}$ are ultraspherical or Gegenbauer polynomials in a new standardization. The lowering and raising operators to the corresponding orthonormalized functions form a simple su(1,1) Lie algebra. This is given in the appendix in sketched form.

Keywords: Jacobi polynomials; Laguerre 2D polynomials; Lie group SU(1,1) and Lie algebra su(1,1); Recurrence relations; Lowering and raising operators; Rodrigues and operational definitions of disc polynomials

1. Introduction

Zernike polynomials are two-dimensional orthogonal polynomials over the unit disc (interior of the unit circle) $0 \leqslant r < 1$ in polar coordinates (r,φ) in the Euclidean plane. They were introduced by Zernike in 1934 [66] when discussing his phase-contrast method in application to circular concave mirrors and were further investigated in [6,37,38,67]. They are taken into account in a few monographs on optics and on geometrical optical imaging, in greater detail by Born and Wolf [7], and

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further in [8,52]. More recent representations of the Zernike polynomials with application to the optical aberrations and with graphical representations are given in [5,27,30,44,54,64,68]. This is carefully summarized in Weisstein's Encyclopedia [55]. The Zernike polynomials are taken into account only in a very few number of mathematical monographs and representations of orthogonal polynomials and special functions. The radial Zernike polynomials were generalized by Myrick [35]. The most natural generalization of the Zernike polynomials is to 2D polynomials which are orthogonal in the unit disc $zz^* < 1$ with weight functions $(1 - zz^*)^\alpha$, where $\alpha > -1$ is a parameter and the special case $\alpha = 0$ is equivalent to the usual Zernike polynomials where, however, different notations and variables are favorable since the variables used by Zernike and in other physical representations are less appropriate for this purpose. The older development of two-variable and multi-variable classical polynomials which began already with Hermite and Didon in nineteenth century and was continued [3] is reviewed by Erdélyi in second volume of the monograph of the Bateman project [19] where one can find the corresponding citations. A comprehensive representation of 2D polynomials with a classification of orthogonal polynomials in restricted domains of the plane and taking into account the disc polynomials among many other polynomials (the first class of the considered seven classes of 2D polynomials) and with discussion of some of their properties was given by Koornwinder [29]. The monograph of Suetin [47] reflects the more general development in the theory of two-variable orthogonal polynomials. A recent monograph of Dunkl and Yuan Xu [18] takes shortly into account the disc polynomials and their 3D analogue, the ball polynomials and gives some historical notes and citations. Disc polynomials are also considered in Vol. 2 (Chapter 11) of the encyclopedic work of Vilenkin and Klimyk about representations of Lie groups and special functions [51] in a representation which in used variables and notations is a little different from that in cited works [18,29] and from that which we use in present paper.

It is known since the first papers [38,66] that the Zernike polynomials are related to Jacobi polynomials, in modern representation to the special case $P_n^{(0,m-n)}(2r^2 - 1)$ of the polynomials $P_n^{(\alpha,\beta)}(u)$ which were introduced in this now generally adapted form in [48] and are discussed, in particular, in [1,2,19,45,46] (an older form with the notation $G_n(p,q,x)\big(\equiv \frac{n!(q-1)!}{(n+q-1)!} P_n^{(q-1,p-q)}(1 - 2x)\big)$ is used in [9] and applied in the representations in [6,7]). The generalization to disc polynomials is related to the case $z^{m-n}P_n^{(\alpha,m-n)}(2zz^* - 1)$ of Jacobi polynomials with α as parameter and $(m,n) = 0,1,\ldots$ as two indices on equal level [18,29]. Although many relations for disc (including Zernike) polynomials can be obtained from corresponding relations for Jacobi polynomials, there is a great difference between them. Whereas Jacobi polynomials $P_n^{(\alpha,\beta)}(u)$ are 1D polynomials with one variable u and with α and β as two independent parameters, the disc polynomials are 2D polynomials with two independent variables (e.g., $z \equiv x + \mathrm{i}y, z^* \equiv x - \mathrm{i}y$) and one parameter α and this brings many new aspects which are not covered by the theory of Jacobi polynomials.

In Section 2, we define generalized Zernike or disc polynomials $P_{m,n}^{\alpha}(z,z^*)$, $(m,n = 0,1,2,\ldots)$ with a continuous parameter $\alpha > -1$ by their relation to the Jacobi polynomials, where the usual Zernike polynomials correspond to the special case $\alpha = 0$. Our standardization is identical with that in [18]. We introduce them in a representation by a pair of complex conjugate variables (z,z^*) in a more symmetrical form concerning the indices in comparison to the usual form of Zernike polynomials and clarify the relations between the different notations. This introduction is in a close analogy to the introduction of the Laguerre 2D polynomials $L_{m,n}(z,z^*)$ which proved to be very useful for applications in classical and quantum optics [57–61]. We show that there exists a limiting transition to the Laguerre 2D polynomials. In Section 3, we discuss the two-dimensional orthogonality of the disc polynomials over the unit disc $0 \leqslant zz^* < 1$ with weight functions $(1-zz^*)^\alpha$ and define orthonormalized functions. The important expansion of the basic monomials $z^m z^{*n}$ into disc polynomials is found. In Section 4, we derive two independent differential equations for the disc polynomials and disc functions and represent them as eigenvalue problem. In Section 5, we find recurrence relations and by means of them, we introduce in Section 6 the lowering and raising operators for the disc polynomials and disc functions. Their commutators can be closed to a Lie algebra $\mathrm{su}(1,1)\oplus\mathrm{su}(1,1)$ belonging to the Lie group $\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)$ of transformations of the disc polynomials that is discussed with some of its consequences in Section 7. In Section 8, we find an operational representation of the disc polynomials which is alternative to the Rodrigues representation and which is very convenient for the derivation of new generating functions that is made in Section 9. In the appendix, we give in sketched form relations for the 1D analogue of the disc polynomials which are the ultraspherical or Gegenbauer polynomials and show that their consideration under this aspect of analogy reveals some new relations and we show that, in particular, the lowering and raising operators for them form a su(1,1) Lie algebra. In a second appendix, we give the explicit form for initial members of sequences of disc polynomials for some low values of the parameter α.

The generalized Zernike or disc polynomials may find multiple applications in cases when one has to do with given functions over a circular disc and when one wants to make expansions into an orthonormalized set of functions over this disc, in particular, in geometrical and wave optics for systems with circular apertures [5–8,27,30,37,38,44,52,54,64,66–68], for example, in Kirchhoff's diffraction integrals where one has to insert the field and its first derivative within the aperture. We hope that they find new applications in other regions, for example, in quantum optics after the properties of these polynomials became better known than before and are available in a form appropriate for direct use.

2. Definition of generalized Zernike or disc polynomials and some basic properties

We introduce generalized Zernike or disc polynomials with the notation $P^\alpha_{m,n}(z,z^*)$ in representation by a pair of complex conjugate variables ($z\equiv x+\mathrm{i}y\equiv r\mathrm{e}^{\mathrm{i}\varphi}, z^*\equiv x-\mathrm{i}y\equiv r\mathrm{e}^{-\mathrm{i}\varphi}$) and with real parameter α by the following definition (agrees with Dunkl and Xu [18] and up to standardization with Koornwinder [29]; cf. also other representation [51, Vol. 2])

$$\begin{aligned} P^\alpha_{m,n}(z,z^*) &\equiv \frac{n!\alpha!}{(n+\alpha)!}\, z^{m-n} P_n^{(\alpha,m-n)}(2zz^*-1) \\ &= \frac{m!\alpha!}{(m+\alpha)!}\, z^{*n-m} P_m^{(\alpha,n-m)}(2zz^*-1) \\ &= z^m z^{*n}\, {}_2F_1\left(-m,-n;\alpha+1;1-\frac{1}{zz^*}\right), \quad (m,n=0,1,2,\ldots), \end{aligned} \tag{2.1}$$

where $P_n^{(\alpha,\beta)}(u)$ denotes the Jacobi polynomials and ${}_2F_1(a,b;c;x)$ the hypergeometric function (e.g., [2,1,19,45,46,48,51]). This introduction allows us to use a big portion of known relations for the Jacobi polynomials and makes it more easy to derive some basic properties of the polynomials, in particular, their orthogonality in the unit disc with weight $(1-zz^*)^\alpha$ and avoids in this way to solve an orthogonalization procedure of the monomials $z^m z^{*n}$ such as by the Schmidt method. Using the general relation for Jacobi polynomials

$$P_n^{(\alpha,\beta)}(u) = (-1)^n P_n^{(\beta,\alpha)}(-u), \tag{2.2}$$

we can write (2.1) in slightly different forms. The generalized Zernike or disc polynomials are 2D polynomials. The usual Zernike polynomials are related to the special case $\alpha = 0$ of $P_{m,n}^{\alpha}(z,z^*)$ but the exact relations we give at the end of this section after we have developed some basic formulae.

From the following two basic explicit representations of the Jacobi polynomials and relations to the hypergeometric function ${}_2F_1(a,b;c;x)$

$$\begin{aligned} P_n^{(\alpha,\beta)}(u) &= \frac{(-1)^n (n+\beta)!}{n!\beta!}\, {}_2F_1\left(-n, n+\alpha+\beta+1; \beta+1; \frac{u+1}{2}\right) \\ &= \frac{(n+\beta)!}{(n+\alpha+\beta)!} \sum_{j=0}^{n} \frac{(-1)^j (2n+\alpha+\beta-j)!}{j!(n-j)!(n+\beta-j)!} \left(\frac{u+1}{2}\right)^{n-j} \\ &= \left(\frac{u+1}{2}\right)^n \sum_{k=0}^{n} \frac{(n+\alpha)!(n+\beta)!}{k!(n-k)!(n+\beta-k)!(k+\alpha)!} \left(\frac{u-1}{u+1}\right)^k \\ &= \left(\frac{u+1}{2}\right)^n \frac{(n+\alpha)!}{n!\alpha!}\, {}_2F_1\left(-n, -n-\beta; \alpha+1; \frac{u-1}{u+1}\right), \end{aligned} \tag{2.3}$$

we find the first basic explicit representation of the disc polynomials ($\{m,n\} \equiv \mathrm{Min}(m,n)$)

$$P_{m,n}^{\alpha}(z,z^*) = \frac{m!n!\alpha!}{(m+\alpha)!(n+\alpha)!} \sum_{j=0}^{\{m,n\}} \frac{(-1)^j (m+n+\alpha-j)!}{j!(m-j)!(n-j)!} z^{m-j} z^{*n-j}, \tag{2.4}$$

and the second basic explicit representation

$$P_{m,n}^{\alpha}(z,z^*) = \sum_{k=0}^{\{m,n\}} \frac{(-1)^k m!n!\alpha!}{k!(m-k)!(n-k)!(k+\alpha)!} (1-zz^*)^k z^{m-k} z^{*n-k}. \tag{2.5}$$

We see that $P_{m,n}^{\alpha}(z,z^*)$ are polynomials of (z,z^*) of degree $m+n$. Both given explicit relations are important and sometimes it is easier to derive a formula from (2.4) and sometimes from (2.5).

Representation (2.5) is well suited to verify the following formula of the Rodrigues type for the disc polynomials which may also serve as definition of these polynomials [29]

$$P_{m,n}^{\alpha}(z,z^*) = \frac{(-1)^{m+n}\alpha!}{(m+n+\alpha)!} \frac{1}{(1-zz^*)^{\alpha}} \frac{\partial^{m+n}}{\partial z^{*m} \partial z^n} (1-zz^*)^{m+n+\alpha}. \tag{2.6}$$

For verification, we form first the derivatives with respect to variable z^* and apply then a known formula for the transformation of $(\partial^n/\partial z^n) z^m$ to "normal ordering" (all powers of z are in front of powers of $\partial/\partial z$) which can easily be proved by complete induction or by Leibniz product rule and it follows

$$\begin{aligned} P_{m,n}^{\alpha}(z,z^*) &= \frac{(-1)^n \alpha!}{(n+\alpha)!} \frac{1}{(1-zz^*)^{\alpha}} \frac{\partial^n}{\partial z^n} z^m (1-zz^*)^{n+\alpha} \\ &= \frac{(-1)^n \alpha!}{(n+\alpha)!(1-zz^*)^{\alpha}} \sum_{k=0}^{\{m,n\}} \frac{m!n!}{k!(m-k)!(n-k)!} z^{m-k} \frac{\partial^{n-k}}{\partial z^{n-k}} (1-zz^*)^{n+\alpha}. \end{aligned} \tag{2.7}$$

The next step of accomplishing the differentiations with respect to variable z leads immediately to the right-hand side of (2.5) and we did not write it down a second time. Later on, in Section 8, we derive yet an alternative operational definition of the disc polynomials.

We now give some simple properties of the disc polynomials. The following properties of parity and of complex conjugation together with index symmetry of the disc polynomials are immediately to see from (2.4) and (2.5)

$$P_{m,n}^{\alpha}(-z,-z^*) = (-1)^{m+n} P_{m,n}^{\alpha}(z,z^*), \quad (P_{m,n}^{\alpha}(z,z^*))^* = P_{m,n}^{\alpha}(z^*,z) = P_{n,m}^{\alpha}(z,z^*). \tag{2.8}$$

From (2.4) follows for the values of the disc polynomials at the coordinate origin $z = z^* = 0$ ($\delta_{m,n}$ is Kronecker symbol)

$$P_{m,n}^{\alpha}(0,0) = \frac{(-1)^n n! \alpha!}{(n+\alpha)!} \delta_{m,n} \tag{2.9}$$

and from (2.5) for the values on the unit circle $r = 1$

$$P_{m,n}^{\alpha}(e^{i\varphi}, e^{-i\varphi}) = e^{i(m-n)\varphi}, \tag{2.10}$$

which last do not depend on parameter α. However, they depend on the location φ on the unit circle and, therefore, the unit disc cannot be transformed into a sphere for the disc polynomials in such a way that the border of the disc is contracted to one point. Furthermore, we see that the disc polynomials are for different values $m - n$ of the indices orthogonal on the unit circle $r = 1$.

In polar coordinates (r, φ), we find the following representation of $P_{m,n}^{\alpha}(z,z^*)$

$$\begin{aligned} P_{m,n}^{\alpha}(re^{i\varphi}, re^{-i\varphi}) &= e^{i(m-n)\varphi} P_{m,n}^{\alpha}(r,r), \\ P_{m,n}^{\alpha}(r,r) &\equiv \frac{(m+\alpha)!}{m!} r^{m-n} P_n^{(\alpha,m-n)}(2r^2-1) \\ &= \frac{m!n!\alpha!}{(m+\alpha)!(n+\alpha)!} \sum_{j=0}^{\{m,n\}} \frac{(-1)^j (m+n+\alpha-j)!}{j!(m-j)!(n-j)!} r^{m+n-2j} \\ &= P_{n,m}^{\alpha}(r,r), \end{aligned} \tag{2.11}$$

where $P_{m,n}^{\alpha}(r,r)$ is a (one-dimensional) polynomial of the variable $r \geqslant 0$ of degree $m + n$ with parameter α which in inversely unique way is related to the Jacobi polynomials. Rotations $R(\varphi_0)$ in the plane around the coordinate origin by an angle φ_0 multiply the polynomials $P_{m,n}^{\alpha}(z,z^*)$ by a factor $e^{-i(m-n)\varphi_0}$. Thus the disc polynomials transform with respect to the rotation group SO(2, R) according to its (one-dimensional) irreducible representations.

From (2.4) follows that in the limiting case $|\alpha| \to \infty$ for fixed (m,n) the disc polynomials make the transition to the monomials $z^m z^{*n}$ in the following way:

$$\lim_{|\alpha|\to\infty} P^{\alpha}_{m,n}(z,z^*) = z^m z^{*n}. \tag{2.12}$$

We mention here a striking analogy of formulae (2.1) and (2.4) for the disc polynomials to corresponding formulae for the special Laguerre 2D polynomials $L_{m,n}(z,z^*)$ [57–61] (called two-variable Hermite polynomials in [21–23]) which are explicitly

$$\begin{aligned} L_{m,n}(z,z^*) &= (-1)^n n! z^{m-n} L_n^{m-n}(zz^*) \\ &= (-1)^m m! z^{*n-m} L_m^{n-m}(zz^*) \\ &= \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m! n!}{j!(m-j)!(n-j)!} z^{m-j} z^{*n-j}, \end{aligned} \tag{2.13}$$

where $L_n^{\nu}(u)$ denotes the generalized Laguerre polynomials. The general Laguerre 2D polynomials contain yet a general 2D matrix U as parameter which becomes the identity matrix I for the special Laguerre 2D polynomials in (2.13) (for related but not identical definitions of two-variable Hermite polynomials; see [10,11,14–17,19,31,49]). Using the explicit representation (2.4) for the disc polynomials, we can prove the following limiting transition to the special Laguerre 2D polynomials (2.13)

$$L_{m,n}(z,z^*) = \lim_{|\alpha|\to\infty} \left(\sqrt{\alpha}\right)^{m+n} P^{\alpha}_{m,n}\left(\frac{z}{\sqrt{\alpha}}, \frac{z^*}{\sqrt{\alpha}}\right). \tag{2.14}$$

This relation possesses a group-theoretical background and corresponds to the transition from the group $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$ to the Heisenberg–Weyl group $W(1,R) \times W(1,R)$ of two harmonic oscillator modes in the sense of a group contraction of İnönü and Wigner [26,65] (see Section 7).

By comparison of (2.4) or of other formulae with corresponding formulae for the usual Zernike polynomials, we find the following relation

$$P^0_{m,n}(r\mathrm{e}^{\mathrm{i}\varphi}, r\mathrm{e}^{-\mathrm{i}\varphi}) \equiv \mathrm{e}^{\mathrm{i}(m-n)\varphi} R^{|m-n|}_{m+n}(r), \tag{2.15}$$

where $R^{|m-n|}_{m+n}(r)$ are the (one-dimensional) radial Zernike polynomials in usual notation [6,55,66] (corresponding to our $P^0_{m,n}(r,r)$ in (2.11)). We see here that the difference $k-l$ of the lower and upper indices in the radial Zernike polynomials $R^l_k(r)$ is restricted to even integers that is a certain inconvenience of this notation.

3. Orthogonality and completeness of disc polynomials on the unit disc

From the known orthogonality of the Jacobi polynomials (e.g., [1,19,45,48])

$$\int_{-1}^{+1} \mathrm{d}u (1-u)^{\alpha} (1+u)^{\beta} P_l^{(\alpha,\beta)}(u) P_n^{(\alpha,\beta)}(u) = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{(n+\alpha)!(n+\beta)!}{n!(n+\alpha+\beta)!} \delta_{l,n}, \tag{3.1}$$

using polar coordinates (r,φ) and transformed variable $u = 2r^2 - 1$ and with $\beta = m - n$, we find the following orthogonality of the one-dimensional polynomials $P^{\alpha}_{m,n}(r,r)$ on the unit radius $0 \leqslant r \leqslant 1$

$$\int_0^1 \mathrm{d}r\, r(1-r^2)^\alpha P^\alpha_{l+m-n,l}(r,r) P^\alpha_{m,n}(r,r) = \frac{m!\alpha!n!\alpha!}{2(m+n+\alpha+1)(m+\alpha)!(n+\alpha)!}\,\delta_{l,n}. \tag{3.2}$$

Such a one-dimensional orthogonality exists only if the difference $k-l$ and $m-n$ of the lower indices in the product of the two one-dimensional polynomials $P^\alpha_{k,l}(r,r)$ and $P^\alpha_{m,n}(r,r)$ are equal and if the upper indices (parameter α) are also equal. Together with the orthogonality of the phase functions

$$\int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{e}^{\mathrm{i}(l-k+m-n)\varphi} = 2\pi\delta_{l+m,k+n}, \tag{3.3}$$

this leads to the following basic orthogonality of the disc polynomials in the unit disc $zz^* \leqslant 1$ ($\frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}z^* = r\,\mathrm{d}r \wedge \varphi = \mathrm{d}x \wedge \mathrm{d}y$ is the area element of the plane)

$$\int_{zz^* \leqslant 1} \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}z^* (1-zz^*)^\alpha (P^\alpha_{k,l}(z,z^*))^* P^\alpha_{m,n}(z,z^*)$$
$$= \frac{\pi m!\alpha!n!\alpha!}{(m+n+\alpha+1)(m+\alpha)!(n+\alpha)!}\,\delta_{k,m}\delta_{l,n}. \tag{3.4}$$

We see that $(1-zz^*)^\alpha = (1-r^2)^\alpha$ is the weight function of the orthogonality relation on the unit disc. The integrals with such weight functions over polynomials within the unit disc converge in usual sense only for $\alpha > -1$.

Relation (3.4) and its special cases $k=m, l=n$ suggest to introduce generalized Zernike or disc functions $p^\alpha_{m,n}(z,z^*)$ in addition to the polynomials by

$$p^\alpha_{m,n}(z,z^*) \equiv \frac{1}{\alpha!}\sqrt{\frac{(m+n+\alpha+1)(m+\alpha)!(n+\alpha)!}{\pi m!n!}}(1-zz^*)^{\alpha/2} P^\alpha_{m,n}(z,z^*). \tag{3.5}$$

These functions are orthonormalized according to

$$\langle p^\alpha_{k,l} | p^\alpha_{m,n}\rangle \equiv \int_{zz^* \leqslant 1} \frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}z^* (p^\alpha_{k,l}(z,z^*))^* p^\alpha_{m,n}(z,z^*) = \delta_{k,m}\delta_{l,n}, \quad \alpha > -1. \tag{3.6}$$

The left-hand side possesses the form of a scalar product in a Hilbert space.

The sets of disc polynomials $P^\alpha_{m,n}(z,z^*)$, $(m,n = 0,1,2,\ldots)$ with fixed α are complete on the unit disc. This follows from Weierstrass approximation theorem (e.g., [48]) or, for example, from their continuous deformation by means of the parameter α which according to (2.12) leads to the complete set of polynomials $z^m z^{*n}$. Together with their orthogonality (3.6) this leads to the completeness relation

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p^\alpha_{m,n}(z,z^*)(p^\alpha_{m,n}(z',z'^*))^*$$
$$= \frac{1}{\pi}\left(\sqrt{(1-zz^*)(1-z'z'^*)}\right)^\alpha$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (m+n+\alpha+1)\,\frac{(m+\alpha)!(n+\alpha)!}{m!\alpha!n\alpha!}\,P^\alpha_{m,n}(z,z^*)(P^\alpha_{m,n}(z',z'^*))^*$$
$$= \delta(z-z', z^*-z'^*), \quad |z|, |z'| < 1, \tag{3.7}$$

where $\delta(z,z^*)$ denotes the two-dimensional delta function over the unit disc (identity operator in Hilbert space of disc functions). Therefore, a function $(1-zz^*)^{\beta/2}f(z,z^*)$ over the unit disc can be expanded into a series over the disc polynomials or functions according to

$$(1-zz^*)^{\beta/2}f(z,z^*)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{}_{\beta}f_{m,n}^{\alpha}\,p_{m,n}^{\alpha}(z,z^*),\quad (zz^*<1), \tag{3.8}$$

with coefficients ${}_{\beta}f_{m,n}^{\alpha}$ given by

$${}_{\beta}f_{m,n}^{\alpha}=\int_{zz^*\leqslant 1}\frac{\mathrm{i}}{2}\,\mathrm{d}z\wedge\mathrm{d}z^*(p_{m,n}^{\alpha}(z,z^*))^*(1-zz^*)^{\beta/2}f(z,z^*), \tag{3.9}$$

where the parameters α and β can be chosen arbitrarily $(-1<\alpha<+\infty)$. For $\beta=\alpha$, we obtain the expansion of the function $f(z,z^*)$ into disc polynomials, for $\beta=0$ into disc functions and for $\beta=-\alpha$ into the dual (or adjoint) functions to disc polynomials.

As an important example of the application of (3.8) and (3.9), we have calculated the expansion of $z^m z^{*n}$ into disc polynomials with the result

$$z^m z^{*n}=\frac{m!n!}{\alpha!}\sum_{j=0}^{\{m,n\}}\frac{(m+\alpha-j)!(n+\alpha-j)!(m+n+1+\alpha-2j)}{j!(m-j)!(n-j)!(m+n+1+\alpha-j)!}P_{m-j,n-j}^{\alpha}(z,z^*). \tag{3.10}$$

This relation gives at once the inversion of (2.4). In particular, for $\alpha=0$ that means for the usual Zernike polynomials it takes on the simple form

$$z^m z^{*n}=m!n!\sum_{j=0}^{\{m,n\}}\frac{m+n+1-2j}{j!(m+n+1-j)!}P_{m-j,n-j}^{0}(z,z^*). \tag{3.11}$$

The use of the Taylor series expansion of a function $f(z,z^*)$ together with (3.10) is another possibility to obtain the expansion of $f(z,z^*)$ into disc polynomials.

The introduction of orthonormalized disc functions is in analogy to the introduction of orthonormalized Hermite 2D and Laguerre 2D functions in addition to the corresponding polynomials [57,59–61]. Using the limiting transition (2.14) and the limiting transition $k\to\infty$ in the expansion $(1-(x/k))^k=\mathrm{e}^{-x}(1-(x^2/2k)-(x^3/3k^2)+\cdots)$ we obtain the Laguerre 2D functions $l_{m,n}(z,z^*)$ by

$$\begin{aligned}l_{m,n}(z,z^*)&\equiv\frac{1}{\sqrt{\pi}}\exp\left(-\frac{zz^*}{2}\right)\frac{1}{\sqrt{m!n!}}L_{m,n}(z,z^*)\\&=\lim_{|\alpha|\to\infty}\frac{1}{\sqrt{\alpha}}\,p_{m,n}^{\alpha}\left(\frac{z}{\sqrt{\alpha}},\frac{z^*}{\sqrt{\alpha}}\right).\end{aligned} \tag{3.12}$$

Thus the Laguerre 2D functions are a limiting case of scaled disc functions.

4. Differential equations for disc polynomials and disc functions

In the following we need some transformation relations between the pair of complex conjugated coordinates (z,z^*) and polar coordinates (r,φ) and, for convenience, give the basic ones

$$z=r\mathrm{e}^{\mathrm{i}\varphi},\quad z^*=r\mathrm{e}^{-\mathrm{i}\varphi},\quad \frac{\partial}{\partial z}=\frac{\mathrm{e}^{-\mathrm{i}\varphi}}{2}\left(\frac{\partial}{\partial r}-\frac{\mathrm{i}}{r}\frac{\partial}{\partial\varphi}\right),\quad \frac{\partial}{\partial z^*}=\frac{\mathrm{e}^{\mathrm{i}\varphi}}{2}\left(\frac{\partial}{\partial r}+\frac{\mathrm{i}}{r}\frac{\partial}{\partial\varphi}\right),$$

$$r=\sqrt{zz^*},\quad \mathrm{e}^{\mathrm{i}\varphi}=\sqrt{\frac{z}{z^*}},\quad r\frac{\partial}{\partial r}=z\frac{\partial}{\partial z}+z^*\frac{\partial}{\partial z^*},\quad \frac{\partial}{\partial\varphi}=\mathrm{i}\left(z\frac{\partial}{\partial z}-z^*\frac{\partial}{\partial z^*}\right). \tag{4.1}$$

The two-dimensional Laplace operator possesses the following form in the two representations:

$$4\frac{\partial^2}{\partial z\partial z^*}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}=\frac{1}{r^2}\left\{\left(r\frac{\partial}{\partial r}\right)^2+\left(\frac{\partial}{\partial\varphi}\right)^2\right\}. \tag{4.2}$$

For the disc polynomials, one can find two independent differential equations. A first and at once first-order differential equation is obtained from the observation that in polar coordinates (r,φ) the disc polynomials are eigenfunctions of the differentiation operator with respect to the phase φ to eigenvalues $\mathrm{i}(m-n)$ according to

$$\frac{\partial}{\partial\varphi}P^{\alpha}_{m,n}(r\mathrm{e}^{\mathrm{i}\varphi},r\mathrm{e}^{-\mathrm{i}\varphi})=\mathrm{i}(m-n)P^{\alpha}_{m,n}(r\mathrm{e}^{\mathrm{i}\varphi},r\mathrm{e}^{-\mathrm{i}\varphi}), \tag{4.3}$$

or in coordinates (z,z^*)

$$\left(z\frac{\partial}{\partial z}-z^*\frac{\partial}{\partial z^*}\right)P^{\alpha}_{m,n}(z,z^*)=(m-n)P^{\alpha}_{m,n}(z,z^*). \tag{4.4}$$

The same equations are obviously true after substitution of the disc polynomials $P^{\alpha}_{m,n}(z,z^*)$ by the corresponding disc functions $p^{\alpha}_{m,n}(z,z^*)$ defined in (3.5).

A second independent and, in this case, second-order differential equation is obtained from the following known differential equation for the Jacobi polynomials [19,48]

$$\left\{(1-u^2)\frac{\partial^2}{\partial u^2}+(m-n-\alpha-(m-n+\alpha+2)u)\frac{\partial}{\partial u}+n(m+\alpha+1)\right\}P^{(\alpha,m-n)}_n(u)=0. \tag{4.5}$$

With the substitution $u=2r^2-1$ and using the definition (2.1) this leads in polar coordinates to the following differential equation:

$$\left\{(1-r^2)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)-2(1+\alpha)r\frac{\partial}{\partial r}+4mn+2(1+\alpha)(m+n)\right\}$$
$$\times P^{\alpha}_{m,n}(r\mathrm{e}^{\mathrm{i}\varphi},r\mathrm{e}^{-\mathrm{i}\varphi})=0. \tag{4.6}$$

Expressed by the pair of complex conjugate coordinates (z,z^*) and written in a symmetrized form this yields the following differential equation for the disc polynomials $P^{\alpha}_{m,n}(z,z^*)$:

$$\left\{2\left((1-zz^*)\frac{\partial^2}{\partial z\partial z^*}+\frac{\partial^2}{\partial z\partial z^*}(1-zz^*)\right)-\alpha\left(z\frac{\partial}{\partial z}+z^*\frac{\partial}{\partial z^*}+\frac{\partial}{\partial z}z+\frac{\partial}{\partial z^*}z^*\right)\right.$$
$$\left.+4mn+2(1+\alpha)(m+n+1)\right\}P^{\alpha}_{m,n}(z,z^*)=0. \tag{4.7}$$

This equation can be represented as an eigenvalue equation of a differential operator which is a Hermitean (selfadjoint) operator only in case of $\alpha=0$ that means for the usual Zernike polynomials. This also becomes clear from the orthogonality relations (3.4) which show that due to the weight function $(1-zz^*)^{\alpha}$ the adjoint functions to $P^{\alpha}_{m,n}(z,z^*)$ agree with $P^{\alpha}_{m,n}(z,z^*)$ only for $\alpha=0$.

In the following we need, at least, two independent differential equations for the orthonormalized functions $p^{\alpha}_{m,n}(z,z^*)$ defined in (3.5). We write them in form of eigenvalue equations. The first-order differential equation (4.4) transforms into

$$L\,p^{\alpha}_{m,n}(z,z^*)=(m-n)\,p^{\alpha}_{m,n}(z,z^*), \tag{4.8}$$

where the Hermitean operator L is defined by

$$L \equiv z\,\frac{\partial}{\partial z} - z^*\,\frac{\partial}{\partial z^*} = \frac{\partial}{\partial z}\,z - \frac{\partial}{\partial z^*}\,z^*. \tag{4.9}$$

This operator can be considered as the operator of the (orbital) angular momentum. It does not depend on the parameter α. In extensions of the here used two-dimensional Euclidean space to a three-dimensional Euclidean space by a coordinate x_3 it becomes the L_3-component of the (orbital) angular momentum $\boldsymbol{L}$ (the spin angular momentum is vanishing since we work with scalar functions $p^{\alpha}_{m,n}(z,z^*)$).

Using the definition (3.5), we find from Eq. (4.7) the following differential equation for the disc functions $p^{\alpha}_{m,n}(z,z^*)$ written as an eigenvalue equation:

$$G^{\alpha}\,p^{\alpha}_{m,n}(z,z^*) = (2m+1+\alpha)(2n+1+\alpha)\,p^{\alpha}_{m,n}(z,z^*) \tag{4.10}$$

for a Hermitean operator G^{α} depending on parameter α and defined by

$$\begin{aligned} G^{\alpha} &\equiv -2\left((1-zz^*)\,\frac{\partial^2}{\partial z\partial z^*} + \frac{\partial^2}{\partial z\partial z^*}(1-zz^*)\right) - 1 + \frac{\alpha^2}{1-zz^*} \\ &= -4\,\frac{\partial^2}{\partial z\partial z^*} + 2\left(z\,\frac{\partial}{\partial z}\,z^*\,\frac{\partial}{\partial z^*} + \frac{\partial}{\partial z}\,z\,\frac{\partial}{\partial z^*}\,z^*\right) - 1 + \frac{\alpha^2}{1-zz^*}. \end{aligned} \tag{4.11}$$

The eigenvalues $(2m+1+\alpha)(2n+1+\alpha)$ on the right-hand side of (4.10) are for $m \neq n$ twice degenerate since pairs (m,n) and (n,m) lead to the same eigenvalue. Therefore, the real-valued combinations $1/2(p^{\alpha}_{m,n}(z,z^*) + p^{\alpha}_{n,m}(z,z^*))$ and $-\mathrm{i}/2(p^{\alpha}_{m,n}(z,z^*) - p^{\alpha}_{n,m}(z,z^*))$ are also solutions of the eigenvalue equation (4.10) to the same eigenvalue. The usual Zernike polynomials are often considered in these real-valued combinations. Due to our way of first constructing the functions $p^{\alpha}_{m,n}(z,z^*)$ and then determining the differential equations which they obey, we know already the eigenvalues and do not have to determine them.

We can use (4.8) to derive from (4.10) other eigenvalue equations for the disc functions. If we add to Eq. (4.10) on the left-hand side $L^2\,p^{\alpha}_{m,n}(z,z^*)$ and use (4.8), then we obtain the following eigenvalue equation

$$H^{\alpha}\,p^{\alpha}_{m,n}(z,z^*) = (m+n+1+\alpha)^2\,p^{\alpha}_{m,n}(z,z^*) \tag{4.12}$$

for a positive definite Hermitean operator H^{α} defined by

$$\begin{aligned} H^{\alpha} &\equiv G^{\alpha} + L^2 \\ &= -4\,\frac{\partial^2}{\partial z\partial z^*} + \left(z\,\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\,z^*\right)^2 + \frac{\alpha^2}{1-zz^*}. \end{aligned} \tag{4.13}$$

This eigenvalue problem seems to us as particularly attractive since we have in (4.12) a higher degeneracy of the eigenvalues on the right-hand side than in (4.10). The eigenvalues are here $(m+n+1)$-fold degenerate since this equation discriminates the eigenvalues only with respect to the sum $m+n$ of indices in the solutions whereas Eq. (4.8) discriminates with respect to the difference $m-n$ of the indices. Therefore, Eq. (4.12) admits also a great manifold of superpositions of functions $p^{\alpha}_{m,n}(z,z^*)$ with fixed $m+n$ as eigensolutions that may serve as starting point for the introduction of a general 2D matrix for superpositions similar as in case of the general Hermite and Laguerre

2D polynomials [59–61]. Furthermore, (4.12) allows to define the square root $\sqrt{H^\alpha}$ of the operator H^α as a positive definite Hermitean operator according to

$$\sqrt{H^\alpha}\, p^\alpha_{m,n}(z,z^*) = (m+n+1+\alpha)\, p^\alpha_{m,n}(z,z^*), \quad \alpha > -1. \tag{4.14}$$

There are yet other possibilities to formulate the eigenvalue problem. If we add $L^2 p^\alpha_{m,n}(z,z^*)$ to Eq. (4.12), we can remove the mixed terms with the product mn in the eigenvalues and obtain the eigenvalue equation

$$I^\alpha p^\alpha_{m,n}(z,z^*) = 2\left(\left(m+\frac{1+\alpha}{2}\right)^2 + \left(n+\frac{1+\alpha}{2}\right)^2\right) p^\alpha_{m,n}(z,z^*) \tag{4.15}$$

for a Hermitean operator I^α defined by

$$\begin{aligned} I^\alpha &\equiv H^\alpha + L^2 \\ &= -4\frac{\partial^2}{\partial z \partial z^*} + 2\left(z\frac{\partial^2}{\partial z^2}z + z^*\frac{\partial^2}{\partial z^{*2}}z^*\right) + 1 + \frac{\alpha^2}{1-zz^*}. \end{aligned} \tag{4.16}$$

The eigenvalues on the right-hand side of (4.15) consist of two additive parts which only separately depend on m and n but do not possess coupling terms. In Section 7 we find how the operator I^α can be additively decoupled into two parts such that each partial operator has only one of the additive parts of the eigenvalues in (4.15) as eigenvalues.

The eigenvalue equation (4.12) (and also (4.10) and (4.15)) is in some analogy to the two-dimensional stationary Schrödinger equation where H^α corresponds to the Hamilton operator. The first part of the Hermitean operator H^α in (4.13) is the negatively taken two-dimensional Laplace operator $4\partial^2/\partial z\partial z^*$ and is in analogy to the part from kinetic energy in the Schrödinger equation but this operator is here modified by an additional part $(z\partial/\partial z + \partial/\partial z^* z^*)^2$, whereas the second part $\alpha^2/(1-zz^*)$ of this operator with parameter α is in analogy to a potential function in the Schrödinger equation. The mentioned modification of the Laplace operator is of the type of Laplace–Beltrami operators [29]. The eigenvalues in equation (4.12) show that we have a quadratic law of level spacing which in one-dimensional quantum-mechanical potential problems in every case is connected with SU(1,1) symmetry as discussed in [62,63]. Later on, in Section 7, we find that this is also the case for the two-dimensional disc functions which are related to $\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)$ symmetry.

For the limiting transition $\alpha \to \infty$ in (4.12) to the Laguerre 2D functions according to (3.12) it is favorable to bring first the term $\alpha^2 p^\alpha_{m,n}(z,z^*)$ from the right-hand side to the left-hand side and then to join it with the potential term leading to the renormalized potential function $\alpha^2 zz^*/(1-zz^*)$.

5. Recurrence relations for disc polynomials and differentiation

The disc polynomials satisfy some recurrence relations which in connection with differentiations are important for the derivation of lowering and raising operators and, finally, for an alternative derivation of the differential equations for the polynomials.

Most easily from (2.5) but also easily from (2.4) follow the recurrence relations

$$(m-n)P^\alpha_{m,n}(z,z^*) = mzP^\alpha_{m-1,n}(z,z^*) - nz^*P^\alpha_{m,n-1}(z,z^*). \tag{5.1}$$

Furthermore, by means of (2.4) we can prove the following recurrence relations:

$$(m+n+1+\alpha)zP_{m,n}^{\alpha}(z,z^*) = (m+1+\alpha)P_{m+1,n}^{\alpha}(z,z^*) + nP_{m,n-1}^{\alpha}(z,z^*),$$
$$(m+n+1+\alpha)z^*P_{m,n}^{\alpha}(z,z^*) = (n+1+\alpha)P_{m,n+1}^{\alpha}(z,z^*) + mP_{m-1,n}^{\alpha}(z,z^*). \tag{5.2}$$

These are the basic recurrence relations.

We now consider first-order differentiations of the disc polynomials. By simple differentiation of the explicit representation (2.4) we obtain the following relations which raise the upper index together with lowering of a lower index [29]

$$\frac{\partial}{\partial z} P_{m,n}^{\alpha}(z,z^*) = \frac{m(n+1+\alpha)}{1+\alpha} P_{m-1,n}^{\alpha+1}(z,z^*),$$
$$\frac{\partial}{\partial z^*} P_{m,n}^{\alpha}(z,z^*) = \frac{(m+1+\alpha)n}{1+\alpha} P_{m,n-1}^{\alpha+1}(z,z^*). \tag{5.3}$$

In connection with raising operators for the lower indices these relations allow to find an operator for raising the parameter α in unit steps.

The two basic relations for the differentiation of the disc polynomials which can be proved by (2.5) are

$$(m+n+1+\alpha)(1-zz^*)\frac{\partial}{\partial z} P_{m,n}^{\alpha}(z,z^*) = m(n+1+\alpha)(P_{m-1,n}^{\alpha}(z,z^*) - P_{m,n+1}^{\alpha}(z,z^*)),$$
$$(m+n+1+\alpha)(1-zz^*)\frac{\partial}{\partial z^*} P_{m,n}^{\alpha}(z,z^*) = (m+1+\alpha)n(P_{m,n-1}^{\alpha}(z,z^*) - P_{m+1,n}^{\alpha}(z,z^*)). \tag{5.4}$$

A further differential equation which is easily to check by differentiation of the explicit representation (2.4) leads to the already discussed eigenvalue equation (4.4).

6. Lowering and raising operators for disc polynomials and for disc functions

We first consider the lowering and raising of the indices in the set of disc polynomials with α as parameter.

By linear combinations of (5.2) and (5.4), we find the following relations for lowering of the indices of the disc polynomials

$$\left(mz^* + (1-zz^*)\frac{\partial}{\partial z}\right) P_{m,n}^{\alpha}(z,z^*) = mP_{m-1,n}^{\alpha}(z,z^*),$$
$$\left(nz + (1-zz^*)\frac{\partial}{\partial z^*}\right) P_{m,n}^{\alpha}(z,z^*) = nP_{m,n-1}^{\alpha}(z,z^*). \tag{6.1}$$

In a similar way by linear combinations of (5.2) and (5.4), we find the corresponding relations for raising of the indices of the disc polynomials which can be represented in the form

$$\left((m+1+\alpha)z - (1-zz^*)\frac{\partial}{\partial z^*}\right) P_{m,n}^{\alpha}(z,z^*) = (m+1+\alpha)P_{m+1,n}^{\alpha}(z,z^*),$$
$$\left((n+1+\alpha)z^* - (1-zz^*)\frac{\partial}{\partial z}\right) P_{m,n}^{\alpha}(z,z^*) = (n+1+\alpha)P_{m,n+1}^{\alpha}(z,z^*). \tag{6.2}$$

With (6.1) and (6.2), we found the lowering and raising operators of the disc polynomials in a representation where they explicitly depend on the indices of the polynomials onto which they act. This is a certain disadvantage and later we will derive representations of the lowering and raising operators for the disc functions in which they do not exhibit an explicit dependence on the indices of these functions.

Using definition (3.5), we can convert the lowering and raising relations for the disc polynomials $P_{m,n}^{\alpha}(z,z^*)$ into lowering and raising relations for the disc functions $p_{m,n}^{\alpha}(z,z^*)$. Acting in this way, we find from (6.1) for the lowering of the indices of the disc functions

$$\begin{aligned} K_-^{(1)}\, p_{m,n}^{\alpha}(z,z^*) &\equiv \sqrt{\frac{m+n+\alpha}{m+n+1+\alpha}}\left(\left(m+\frac{\alpha}{2}\right)z^* + (1-zz^*)\,\frac{\partial}{\partial z}\right) p_{m,n}^{\alpha}(z,z^*) \\ &= \sqrt{(m+\alpha)\,m}\, p_{m-1,n}^{\alpha}(z,z^*), \\ K_-^{(2)}\, p_{m,n}^{\alpha}(z,z^*) &\equiv \sqrt{\frac{m+n+\alpha}{m+n+1+\alpha}}\left(\left(n+\frac{\alpha}{2}\right)z + (1-zz^*)\,\frac{\partial}{\partial z^*}\right) p_{m,n}^{\alpha}(z,z^*) \\ &= \sqrt{(n+\alpha)\,n}\, p_{m,n-1}^{\alpha}(z,z^*) \end{aligned} \tag{6.3}$$

and from (6.2) for the raising of the indices of the disc functions

$$\begin{aligned} K_+^{(1)}\, p_{m,n}^{\alpha}(z,z^*) &\equiv \sqrt{\frac{m+n+2+\alpha}{m+n+1+\alpha}}\left(\left(m+\frac{\alpha}{2}\right)z - \frac{\partial}{\partial z^*}(1-zz^*)\right) p_{m,n}^{\alpha}(z,z^*) \\ &= \sqrt{(m+1+\alpha)(m+1)}\, p_{m+1,n}^{\alpha}(z,z^*), \\ K_+^{(2)}\, p_{m,n}^{\alpha}(z,z^*) &\equiv \sqrt{\frac{m+n+2+\alpha}{m+n+1+\alpha}}\left(\left(n+\frac{\alpha}{2}\right)z^* - \frac{\partial}{\partial z}\,(1-zz^*)\right) p_{m,n}^{\alpha}(z,z^*) \\ &= \sqrt{(n+1+\alpha)(n+1)}\, p_{m,n+1}^{\alpha}(z,z^*), \end{aligned} \tag{6.4}$$

where we have introduced two lowering operators $K_-^{(1)}$ and $K_-^{(2)}$ and two raising operators $K_+^{(1)}$ and $K_+^{(2)}$ by their action onto the complete set of basis functions $p_{m,n}^{\alpha}(z,z^*)$. The disc functions $p_{0,n}^{\alpha}(z,z^*)$ are completely annihilated by $K_-^{(1)}$ and the disc functions $p_{m,0}^{\alpha}(z,z^*)$ completely by $K_-^{(2)}$.

From the orthonormalization of the basis functions $p_{m,n}^{\alpha}(z,z^*)$ together with (6.3) and (6.4) follows that the lowering and raising operators are Hermitean adjoint to each other as follows:

$$K_{\mp}^{(1)\dagger} = K_{\pm}^{(1)}, \quad K_{\mp}^{(2)\dagger} = K_{\pm}^{(2)}. \tag{6.5}$$

For the commutators of the lowering and raising operators, we find from (6.3) and (6.4)

$$\begin{aligned} [K_-^{(1)},K_+^{(1)}]\, p_{m,n}^{\alpha}(z,z^*) &= 2\left(m+\frac{1+\alpha}{2}\right) p_{m,n}^{\alpha}(z,z^*) \equiv 2K_0^{(1)}\, p_{m,n}^{\alpha}(z,z^*), \\ [K_-^{(2)},K_+^{(2)}]\, p_{m,n}^{\alpha}(z,z^*) &= 2\left(n+\frac{1+\alpha}{2}\right) p_{m,n}^{\alpha}(z,z^*) \equiv 2K_0^{(2)}\, p_{m,n}^{\alpha}(z,z^*), \end{aligned} \tag{6.6}$$

where we have introduced two new Hermitean operators $K_0^{(1)}$ and $K_0^{(2)}$ by their action onto the complete set of basis functions $p_{m,n}^{\alpha}(z,z^*)$. The commutators of the lowering and raising operators with different upper numbers are vanishing

$$[K_-^{(1)},K_+^{(2)}]\,p_{m,n}^{\alpha}(z,z^*)=0,\quad [K_-^{(2)},K_+^{(1)}]\,p_{m,n}^{\alpha}(z,z^*)=0. \tag{6.7}$$

The operators $K_0^{(1)}$ and $K_0^{(2)}$ are, up to the additive part $(1+\alpha)/2$, the number operators of the disc functions which applied to these functions produce the numbers m and n of these functions.

We now derive another form of the lowering and raising relations for the disc functions where the corresponding operators do no more explicitly show a dependence on the indices of the disc functions onto which they act. For this purpose, we act first with the lowering operator $K_-^{(1)}$ onto $p_{m,n}^{\alpha}(z,z^*)$ and then with the raising operator $K_+^{(1)}$ onto the result of $K_-^{(1)}p_{m,n}^{\alpha}(z,z^*)$ and obtain a differential equation which after division by $1-zz^*$ can be written as the following quadratic equation for $(m+\alpha/2)\,p_{m,n}^{\alpha}(z,z^*)$:

$$\begin{aligned}&\left\{\left(m+\frac{\alpha}{2}\right)^2-\left(z\frac{\partial}{\partial z}-\frac{\partial}{\partial z^*}z^*\right)\left(m+\frac{\alpha}{2}\right)+(1-zz^*)\frac{\partial^2}{\partial z\partial z^*}-z\frac{\partial}{\partial z}-\frac{\alpha^2}{4(1-zz^*)}\right\}\\&\quad\times p_{m,n}^{\alpha}(z,z^*)=0.\end{aligned} \tag{6.8}$$

In a similar way acting first with $K_-^{(2)}$ onto $z_{m,n}^{\alpha}(z,z^*)$ and then with $K_+^{(2)}$ onto the result, we obtain after division by $1-zz^*$ the following quadratic equation for $(n+(\alpha/2))\,p_{m,n}^{\alpha}(z,z^*)$:

$$\begin{aligned}&\left\{\left(n+\frac{\alpha}{2}\right)^2-\left(z^*\frac{\partial}{\partial z^*}-\frac{\partial}{\partial z}z\right)\left(n+\frac{\alpha}{2}\right)+(1-zz^*)\frac{\partial^2}{\partial z\partial z^*}-z^*\frac{\partial}{\partial z^*}-\frac{\alpha^2}{4(1-zz^*)}\right\}\\&\quad\times p_{m,n}^{\alpha}(z,z^*)=0.\end{aligned} \tag{6.9}$$

The solution of the quadratic equation (6.8) is

$$\begin{aligned}\left(m+\frac{\alpha}{2}\right)p_{m,n}^{\alpha}(z,z^*)&=\frac{1}{2}\left(z\frac{\partial}{\partial z}-\frac{\partial}{\partial z^*}z^*+\sqrt{H^{\alpha}}\right)p_{m,n}^{\alpha}(z,z^*)\\&=\frac{1}{2}\left(\sqrt{H^{\alpha}}+L-1\right)p_{m,n}^{\alpha}(z,z^*)\end{aligned} \tag{6.10}$$

and the solution of the quadratic equation (6.9) is

$$\begin{aligned}\left(n+\frac{\alpha}{2}\right)p_{m,n}^{\alpha}(z,z^*)&=\frac{1}{2}\left(z^*\frac{\partial}{\partial z^*}-\frac{\partial}{\partial z}z+\sqrt{H^{\alpha}}\right)p_{m,n}^{\alpha}(z,z^*)\\&=\frac{1}{2}\left(\sqrt{H^{\alpha}}-L-1\right)p_{m,n}^{\alpha}(z,z^*),\end{aligned} \tag{6.11}$$

where we have used for abbreviation the definition of the operators H^{α} given in (4.13) and of L given in (4.9). These relations can also be obtained by forming the sum and the difference of the eigenvalue equation for the square root of H^{α} according to (4.14) and the eigenvalue equation (4.8) for L and we see that the positive sign of the square root of H^{α} has to be taken.

Using (6.10), (6.11) and (4.12) we find that the lowering operators $K_-^{(1)}$ and $K_-^{(2)}$ in (6.3) can be represented in the following form:

$$K_-^{(1)} = (H^\alpha)^{1/4}\left\{\frac{z^*}{2}\left((H^\alpha)^{1/2} + z\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}z^*\right) + (1-zz^*)\frac{\partial}{\partial z}\right\}(H^\alpha)^{-1/4},$$

$$K_-^{(2)} = (H^\alpha)^{1/4}\left\{\frac{z}{2}\left((H^\alpha)^{1/2} + z^*\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z}z\right) + (1-zz^*)\frac{\partial}{\partial z^*}\right\}(H^\alpha)^{-1/4}. \tag{6.12}$$

In analogous way, using (6.10), (6.11) and (4.12) we find that the raising operators $K_+^{(1)}$ and $K_+^{(2)}$ in (6.4) can be represented in the following form:

$$K_+^{(1)} = (H^\alpha)^{1/4}\left\{\frac{z}{2}\left((H^\alpha)^{1/2} + z\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}z^*\right) - \frac{\partial}{\partial z^*}(1-zz^*)\right\}(H^\alpha)^{-1/4},$$

$$K_+^{(2)} = (H^\alpha)^{1/4}\left\{\frac{z^*}{2}\left((H^\alpha)^{1/2} + z^*\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z}z\right) - \frac{\partial}{\partial z}(1-zz^*)\right\}(H^\alpha)^{-1/4}. \tag{6.13}$$

Furthermore, by comparison of (6.10) and (6.11) with (6.6), we find that the operators $K_0^{(1)}$ and $K_0^{(2)}$ can be represented by

$$K_0^{(1)} = \frac{1}{2}\left(\sqrt{H^\alpha} + L\right) = K_0^{(1)\dagger}, \quad K_0^{(2)} = \frac{1}{2}\left(\sqrt{H^\alpha} - L\right) = K_0^{(2)\dagger}, \tag{6.14}$$

where the more detailed form can be obtained using the explicit form of the involved operators given in (4.13) and (4.9).

With (6.12) and (6.13) we have found a covariant (or basis-independent) representation of the lowering and raising operators and with (6.14) a covariant representation of the (modified) number operators of disc functions which do not show an explicit dependence on the indices (m,n) of the functions $p_{m,n}^\alpha(z,z^*)$ onto which they act. The lowering operators $K_-^{(1)}$ and $K_-^{(2)}$ and raising operators $K_+^{(1)}$ and $K_+^{(2)}$ and the operators $K_0^{(1)}$ and $K_0^{(2)}$ are nonlinear in the basis operators $\partial/\partial z$ and z and in $\partial/\partial z^*$ and z^* which form two pairs of annihilation and creation operators of a Heisenberg–Weyl algebra (in boson operator representation) according to the commutation relations

$$\left[\frac{\partial}{\partial z}, z\right] = 1, \quad \left[\frac{\partial}{\partial z^*}, z^*\right] = 1. \tag{6.15}$$

This nonlinearity is similar in the form to the nonlinearity which we found for the SU(1,1) lowering and raising operators for one-dimensional quantum-mechanical potential problems with a quadratic law of energy level spacing [62].

The conjugation relations (6.5) allow to obtain in an easy way alternative representations of the lowering and raising operators. We obtain from (6.13) by Hermitean conjugation

$$K_-^{(1)} \equiv (H^\alpha)^{-1/4}\left\{\left((H^\alpha)^{1/2} + z\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}z^*\right)\frac{z^*}{2} + (1-zz^*)\frac{\partial}{\partial z^*}\right\}(H^\alpha)^{1/4},$$

$$K_-^{(2)} \equiv (H^\alpha)^{-1/4}\left\{\left((H^\alpha)^{1/2} + z^*\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z}z\right)\frac{z}{2} + (1-zz^*)\frac{\partial}{\partial z^*}\right\}(H^\alpha)^{1/4}, \tag{6.16}$$

and from (6.3) by Hermitean conjugation

$$K_+^{(1)} \equiv (H^\alpha)^{-1/4} \left\{ \left((H^\alpha)^{1/2} + z\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*} z^* \right) \frac{z}{2} - \frac{\partial}{\partial z^*}(1 - zz^*) \right\} (H^\alpha)^{1/4},$$

$$K_+^{(2)} \equiv (H^\alpha)^{-1/4} \left\{ \left((H^\alpha)^{1/2} + z^*\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z} z \right) \frac{z^*}{2} - \frac{\partial}{\partial z}(1 - zz^*) \right\} (H^\alpha)^{1/4}. \quad (6.17)$$

It is more difficult to derive these relations by applying the commutation relations of $z, \partial/\partial z$ and $z^*, \partial/\partial z^*$ with powers of H^α.

7. Lie algebra su(1, 1) $\oplus$ su(1, 1) of lowering and raising operators to disc functions

It is known that many differential equations and sets of polynomials and special functions as their solutions are related to a symmetry group. There exists an extensive literature since the 1950s about such relations from which we cite here the monographs [25,32–34,39,40,50] and the work of encyclopedic character [51]. The discussion of the algebra of lowering and raising operations for the disc functions $p^\alpha_{m,n}(z,z^*)$ with chosen notations in last section was aimed to a symmetry treatment by a Lie algebra which we will make in this section. We did not find hints in literature about the existence of the following specialized treatment for disc polynomials.

The operators $K_0^{(1)}$ and $K_0^{(2)}$ were introduced in (6.6) by definition as the notation for the halved commutators of the lowering and raising operators that means for $\frac{1}{2}[K_-^{(1)}, K_+^{(1)}]$ and $\frac{1}{2}[K_-^{(2)}, K_+^{(2)}]$, respectively. If we now calculate the commutators of $K_0^{(1)}$ and $K_0^{(2)}$ with the lowering and raising operators, we find that they lead back to the last operators and the algebra of commutators is closed and becomes a Lie algebra. Altogether, we find the following commutation relations (most easily from representations (6.3), (6.4) and (6.6))

$$[K_-^{(\mu)}, K_+^{(v)}] = 2K_0^{(v)}\delta^{(\mu,v)}, \quad [K_0^{(\mu)}, K_\mp^{(v)}] = \mp K_\mp^{(v)}\delta^{(\mu,v)},$$

$$[K_-^{(1)}, K_-^{(2)}] = [K_+^{(1)}, K_+^{(2)}] = [K_0^{(1)}, K_0^{(2)}] = 0, \quad (\mu, v = 1, 2.). \quad (7.1)$$

All commutators with different upper numbers $\mu \neq v$ are vanishing. Thus the six-dimensional Lie algebra with the 6 basis operators $(K_-^{(1)}, K_-^{(2)}, K_+^{(1)}, K_+^{(2)}, K_0^{(1)}, K_0^{(2)})$ decomposes into two uncoupled three-dimensional Lie algebras of rank 1 with the basis operators $(K_-^{(1)}, K_+^{(1)}, K_0^{(1)})$ and $(K_-^{(2)}, K_+^{(2)}, K_0^{(2)})$, respectively. Due to well-known commutation relations, both three-dimensional subalgebras are su(1,1) Lie algebras belonging to the Lie group SU(1,1) and we have the structure $\mathrm{su}(1,1) \oplus \mathrm{su}(1,1)$ of the whole Lie algebra belonging to the direct product $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$ of the corresponding whole Lie group. The two commuting operators $(K_0^{(1)}, K_0^{(2)})$ form a basis of the (two-dimensional) Cartan subalgebra.

Each set of disc functions $p^\alpha_{m,n}(z,z^*)$ with α as a parameter realizes a unitary irreducible representation (irrep) of the group $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$. To see this, we consider a simple SU(1,1) group with the basis operators (K_-, K_+, K_0) of the Lie algebra (we omit now upper numbers) and with the commutation relations

$$[K_-, K_+] = 2K_0, \quad [K_0, K_-] = -K_-, \quad [K_0, K_+] = +K_+. \quad (7.2)$$

As is well known (e.g., [42,43,53]), the unitary irreps of SU(1,1) with a state of lowest weight which is annihilated by the operator K_- are infinite-dimensional and can be uniquely indicated by a label $k>0$ according to their action onto an orthonormalized set of basis states $|k,n\rangle$ which is diagonal for the Hermitean operator K_0 (see also, for example, [62,63])

$$\begin{aligned} K_-|k,n\rangle &= \sqrt{(n-1+2k)n}\,|k,n-1\rangle, \quad \Rightarrow \quad K_-|k,0\rangle = 0,\\ K_+|k,n\rangle &= \sqrt{(n+2k)(n+1)}\,|k,n+1\rangle,\\ K_0|k,n\rangle &= (n+k)\,|k,n\rangle, \quad \langle k,m|k,n\rangle = \delta_{m,n}, \quad (m,n=0,1,\ldots). \end{aligned} \tag{7.3}$$

We apply here Dirac's notation of "ket" and "bra" vectors for abstract states and costates of a Hilbert space. The irrep (7.3) can easily be constructed from existence of a state $|k,0\rangle$ by supposing $K_0|k,0\rangle = k|k,0\rangle$ and $K_-|k,0\rangle = 0$ and by applying repeatedly to it the raising operator K_+ under consideration of the commutation relations. The Casimir operator C which commutes with all operators of the Lie algebra is for an irrep proportional to the identity operator I, concretely

$$C \equiv (K_0)^2 - \frac{1}{2}(K_-K_+ + K_+K_-) = k(k-1)I, \quad I \equiv \sum_{k=0}^{\infty} |k,n\rangle\langle k,n|. \tag{7.4}$$

The disc functions $p^{\alpha}_{m,n}(z,z^*)$ transform according to the product of two infinite-dimensional irreps of SU(1,1) and by comparison of relations (6.3), (6.4) and (6.6) for upper index (1) and for upper index (2) with (7.3), we find that both subsystems belong to irreps of SU(1,1) with the same label k according to

$$k^{(1)} = k^{(2)} = \frac{\alpha+1}{2}. \tag{7.5}$$

There exist two Casimir operators as which can be taken the Casimir operators $C^{(1)}$ and $C^{(2)}$ of the two subsystems for which we find

$$C^{(\mu)} \equiv (K_0^{(\mu)})^2 - \frac{1}{2}(K_-^{(\mu)}K_+^{(\mu)} + K_+^{(\mu)}K_-^{(\mu)}) = \left(\frac{\alpha+1}{2}\right)\left(\frac{\alpha-1}{2}\right)I, \quad (\mu=1,2), \tag{7.6}$$

where $I = I^{(1)} \times I^{(2)}$ is the identity operator for the whole system. The Lie algebra operators depend here on the parameter α as, in general, they depend on label k which we did not show in our notations to avoid overloading of them with symbols. Contrary to the abstract form of the irrep in (7.3), we work in case of the disc function with a concrete realization by a set of functions $p^{\alpha}_{m,n}(z,z^*)$ which may be obtained from corresponding abstract state vectors $|p^{\alpha}_{m,n}\rangle$ as scalar products $p^{\alpha}_{m,n}(z,z^*) \equiv \langle z,z^*|p^{\alpha}_{m,n}\rangle$, where $|z,z^*\rangle$ is a complete continuous basis generated by the common eigenstates of two Hermitean operators of a two-mode Heisenberg–Weyl algebra and orthonormalized by means of the (here two-dimensional) delta function (similar to canonical operators Q or P in quantum mechanics of one degree of freedom). Furthermore, with (6.12)–(6.14), we have given the operators of the Lie algebra $\mathrm{su}(1,1) \oplus \mathrm{su}(1,1)$ in form of functions of multiplication and differentiation operators.

The 6 operators $(K_-^{(1)}, K_+^{(1)}, K_0^{(1)}, K_-^{(2)}, K_+^{(2)}, K_0^{(2)})$ of the Lie algebra are basic for the transformations of disc polynomials $p^{\alpha}_{m,n}(z,z^*)$ with fixed parameter α. Other operators which play a role in this theory should be expressible by them. In particular, we find from (6.14) that the operator L can be expressed as a linear combination of the operators $K_0^{(1)}$ and $K_0^{(1)}$ in the following way:

$$L = K_0^{(1)} - K_0^{(2)} \tag{7.7}$$

and, therefore, belongs to the Lie algebra $\mathrm{su}(1,1) \oplus \mathrm{su}(1,1)$ itself. From the same relations (6.14), we find for the operator H^α

$$\begin{aligned} H^\alpha &= (K_0^{(1)} + K_0^{(2)})^2 \\ &= \frac{1}{2}\{K_-^{(1)}K_+^{(1)} + K_+^{(1)}K_-^{(1)} + K_-^{(2)}K_+^{(2)} + K_+^{(2)}K_-^{(2)} + 4K_0^{(1)}K_0^{(2)} + (\alpha^2 - 1)I\}, \end{aligned} \tag{7.8}$$

where we used the representation (7.6) of the Casimir operators. Contrary to H^α, its square root

$$\sqrt{H^\alpha} = K_0^{(1)} + K_0^{(2)} \tag{7.9}$$

belongs to the Lie algebra $\mathrm{su}(1,1) \oplus \mathrm{su}(1,1)$. The operators G^α and I^α in (4.11) and (4.16) can be expressed by

$$\begin{aligned} G^\alpha &= 4K_0^{(1)}K_0^{(2)}, \\ I^\alpha &= 2((K_0^{(1)})^2 + (K_0^{(2)})^2) \\ &= K_-^{(1)}K_+^{(1)} + K_+^{(1)}K_-^{(1)} + K_-^{(2)}K_+^{(2)} + K_+^{(2)}K_-^{(2)} + (\alpha^2 - 1)I. \end{aligned} \tag{7.10}$$

The operator I^α consists of two additive parts which belong to the two subsystems with upper indices (1) and (2). The operators G^α, H^α and I^α do not belong to the Lie algebra of the disc polynomials but are quadratic combinations of the operators of this Lie algebra.

The knowledge that the disc functions $p_{m,n}^\alpha(z,z^*)$ realize certain unitary irreps of the Lie group $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$ allows to overtake and to apply some well-known constructions from group theory. For example, the functions $p_{m,n}^\alpha(z,z^*)$ can be generated from the ground (or vacuum) function $p_{0,0}^\alpha(z,z^*)$ by applying repeatedly the raising operators $K_+^{(1)}$ and $K_+^{(2)}$ onto it as follows:

$$p_{m,n}^\alpha(z,z^*) = \frac{\alpha!}{\sqrt{m!(m+\alpha)!n!(n+\alpha)!}}(K_+^{(1)})^m(K_+^{(2)})^n p_{0,0}^\alpha(z,z^*). \tag{7.11}$$

One can also form interesting superpositions of the functions $p_{m,n}^\alpha(z,z^*)$ similar to the different SU(1,1) coherent states in quantum mechanics and can discuss their properties [4,28,41,43,53] and, e.g., [63]. This is connected with the transition from the Lie algebra $\mathrm{su}(1,1) \oplus \mathrm{su}(1,1)$ to the Lie group by exponential mapping and includes the discussion of disentanglement relations of the group operators. This is a greater complex of problems which we will not discuss here.

The transition $k \to \infty$ of the label of the unitary irreps of SU(1,1) is a group contraction in the sense of İnönü and Wigner [26] (and, e.g., [65]) and leads to the Heisenberg–Weyl group $W(1,R)$ of a pair $(a,a^\dagger)$ of boson annihilation and creation operators $(K_-/\sqrt{2k} \to a, K_+/\sqrt{2k} \to a^\dagger, K_0/k \to I)$ [62,63]. According to (7.5), both labels $k^{(1)}$ and $k^{(2)}$ of the unitary irreps of $\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)$ for the disc polynomials are the same and the transition $k \to \infty$ means $\alpha \to \infty$. The group $\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)$ makes then the contraction to a two-mode Heisenberg–Weyl group $W(2,R) \equiv W(1,R) \times W(1,R)$. The disc polynomials $P_{m,n}^\alpha(z,z^*)$ and disc function $p_{m,n}^\alpha(z,z^*)$ make in this limiting procedure the transition to (special) Laguerre 2D polynomials $L_{m,n}(z,z^*)$ and (special) Laguerre 2D functions $l_{m,n}(z,z^*)$, correspondingly, according to (2.14) and (3.12). The relation of the Laguerre 2D functions to a two-mode Heisenberg–Weyl group is discussed in [57,59,60]. The Laguerre 2D polynomials are expressible by superpositions of products of Hermite polynomials and their relation to a one-mode Heisenberg–Weyl group is well known from quantum mechanics of the harmonic oscillator.

8. Alternative operational representation of disc polynomials

For the purpose to derive an alternative operational definition of the disc polynomials, we make the following transformation of the complex variables (z, z^*) into new complex variables (w, w^*) according to

$$(z, z^*) = \left(\frac{w}{\sqrt{1+ww^*}}, \frac{w^*}{\sqrt{1+ww^*}} \right) \leftrightarrow (w, w^*) = \left(\frac{z}{\sqrt{1-zz^*}}, \frac{z^*}{\sqrt{1-zz^*}} \right),$$

$$(1-zz^*)(1+ww^*) = 1, \quad \mathrm{d}w \wedge \mathrm{d}w^* = \frac{\mathrm{d}z \wedge \mathrm{d}z^*}{(1-zz^*)^2}, \quad \mathrm{d}z \wedge \mathrm{d}z^* = \frac{\mathrm{d}w \wedge \mathrm{d}w^*}{(1+ww^*)^2}. \tag{8.1}$$

The unit disc is then transformed into an entire Euclidean plane. In polar coordinates (r, φ), it is a transformation to new polar coordinates (s, χ) with untransformed angle $\varphi = \chi$ and with

$$(r, \varphi) \quad \leftrightarrow \quad (s, \chi), \quad r = \frac{s}{\sqrt{1+s^2}}, \quad s = \frac{r}{\sqrt{1-r^2}}, \quad \varphi = \chi. \tag{8.2}$$

It transforms the disc polynomials in the following way:

$$\begin{aligned} &P^{\alpha}_{m,n}\left(\frac{w}{\sqrt{1+ww^*}}, \frac{w^*}{\sqrt{1+ww^*}} \right) \\ &= \frac{\alpha!}{\left(\sqrt{1+ww^*}\right)^{m+n}} \sum_{k=0}^{\{m,n\}} \frac{(-1)^k m! n!}{k!(m-k)!(n-k)!(k+\alpha)!} w^{m-k} w^{*n-k} \\ &= \frac{1}{\left(\sqrt{1+ww^*}\right)^{m+n}} \sum_{k=0}^{\infty} \frac{(-1)^k \alpha!}{k!(k+\alpha)!} \frac{\partial^{2k}}{\partial w^k \partial w^{*k}} w^m w^{*n}, \end{aligned} \tag{8.3}$$

where it is possible as done to extend the upper summation limit to infinity. Using now the Taylor series of the functions $\alpha! 2^{\alpha} J_{\alpha}(x)/x^{\alpha}$ where $J_{\alpha}(x)$ denotes the Bessel functions

$$\begin{aligned} \frac{\alpha! 2^{\alpha} J_{\alpha}(x)}{x^{\alpha}} &= \sum_{k=0}^{\infty} \frac{(-1)^k \alpha!}{k!(k+\alpha)!} \left(\frac{x}{2}\right)^{2k} \\ &= 1 - \frac{x^2}{4(\alpha+1)} + \frac{x^4}{32(\alpha+2)(\alpha+1)} - \cdots, \end{aligned} \tag{8.4}$$

we can represent (8.3) as follows:

$$P^{\alpha}_{m,n}\left(\frac{w}{\sqrt{1+ww^*}}, \frac{w^*}{\sqrt{1+ww^*}} \right) = \frac{1}{\left(\sqrt{1+ww^*}\right)^{m+n}} \frac{\alpha! 2^{\alpha} J_{\alpha}\left(2\sqrt{\partial^2/\partial w \partial w^*}\right)}{\left(2\sqrt{\partial^2/\partial w \partial w^*}\right)^{\alpha}} w^m w^{*n}. \tag{8.5}$$

With this expression, we have found in transformed variables (w, w^*) an operator of the form $\alpha! 2^{\alpha} J_{\alpha}\left(\sqrt{u}\right) / \left(\sqrt{u}\right)^{\alpha}$ with $u = 4\partial^2/\partial w \partial w^*$ the two-dimensional Laplace operator in the argument which generates from the monomials $w^m w^{*n}$ a principal part of the disc polynomials. The main advantage of this operator is that it does not depend on the indices (m, n) of the disc polynomials and depends only on the parameter α. Analogous operational definitions are known for usual Hermite and Laguerre polynomials [12,13,20,24,36,56] and for corresponding 2D polynomials [57–59] and by discussed limiting transition we come from the operational definition (8.5) of disc polynomials to the operational definition of Laguerre 2D polynomials.

The inverse transformation to primary variables (z, z^*) uses (8.1) from which follows:

$$\frac{\partial}{\partial z} = \frac{\sqrt{1+ww^*}}{2}\left((2+ww^*)\frac{\partial}{\partial w} + w^{*2}\frac{\partial}{\partial w^*}\right),$$

$$\frac{\partial}{\partial z^*} = \frac{\sqrt{1+ww^*}}{2}\left(w^2\frac{\partial}{\partial w} + (2+ww^*)\frac{\partial}{\partial w^*}\right),$$

$$\frac{\partial}{\partial w} = \frac{\sqrt{1-zz^*}}{2}\left((2-zz^*)\frac{\partial}{\partial z} - z^{*2}\frac{\partial}{\partial z^*}\right),$$

$$\frac{\partial}{\partial w^*} = \frac{\sqrt{1-zz^*}}{2}\left(-z^2\frac{\partial}{\partial z} + (2-zz^*)\frac{\partial}{\partial z^*}\right) \tag{8.6}$$

and therefore

$$4\frac{\partial^2}{\partial w \partial w^*} = \frac{1-zz^*}{zz^*}\left\{\left((1-zz^*)\left(z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*}\right)\right)^2 - \left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right)^2\right\}. \tag{8.7}$$

The transformation of the differential equation (4.12) can be found using the identity

$$-4\frac{\partial^2}{\partial z \partial z^*} + \left(z\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}z^*\right)^2 = -(1+ww^*)\left\{4\frac{\partial^2}{\partial w \partial w^*} + \left(w\frac{\partial}{\partial w} + w^*\frac{\partial}{\partial w^*}\right)^2\right\} + 1. \tag{8.8}$$

In the next section, using the operational definition (8.5) of the disc polynomials, we derive new generating functions for these polynomials.

9. A class of generating functions for disc polynomials

The operational representation (8.5) is very convenient for the derivation of a certain class of generating functions for the disc polynomials. We choose an arbitrary function $f(z, z^*)$ of the two variables (z, z^*) and denote its mth derivatives with respect to variable z and nth derivative with respect to variable z^* at $z = z^* = 0$ by $f^{(m,n)}(0,0)$. Then we obtain from (8.5)

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{f^{(m,n)}(0,0)s^m s^{*n}}{m!n!}\left(\sqrt{1+ww^*}\right)^{m+n} P_{m,n}^{\alpha}\left(\frac{w}{\sqrt{1+ww^*}}, \frac{w^*}{\sqrt{1+ww^*}}\right)$$

$$= \frac{\alpha! J_\alpha\left(2\sqrt{\frac{\partial^2}{\partial w \partial w^*}}\right)}{\left(\sqrt{\frac{\partial^2}{\partial w \partial w^*}}\right)^{\alpha}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f^{(m,n)}(0,0)\frac{(sw)^m (s^* w^*)^n}{m!n!}$$

$$= \frac{\alpha! J_\alpha\left(2\sqrt{\frac{\partial^2}{\partial w \partial w^*}}\right)}{\left(\sqrt{\frac{\partial^2}{\partial w \partial w^*}}\right)^\alpha} f(sw, s^* w^*). \tag{9.1}$$

The application of a function of the two-dimensional Laplace operator to a function $f(sw, s^*w^*)$ becomes very simple if $f(z,z^*)$ is an eigenfunction of this operator to eigenvalues $-\kappa^2$ that means if it is a solution of the two-dimensional Helmholtz equation (we denote such functions by $f_{\mathrm{H}}(z,z^*)$)

$$\left(4\frac{\partial^2}{\partial z \partial z^*} + \kappa^2\right) f_{\mathrm{H}}(z,z^*) = 0. \tag{9.2}$$

In this case, we obtain from (9.1)

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{f_{\mathrm{H}}^{(m,n)}(0,0) s^m s^{*n}}{m!n!} \left(\sqrt{1+ww^*}\right)^{m+n} P_{m,n}^{\alpha}\left(\frac{w}{\sqrt{1+ww^*}}, \frac{w^*}{\sqrt{1+ww^*}}\right)$$
$$= \frac{\alpha! 2^\alpha I_\alpha\left(\kappa\sqrt{ss^*}\right)}{\left(\kappa\sqrt{ss^*}\right)^\alpha} f_{\mathrm{H}}(sw, s^*w^*), \tag{9.3}$$

where $I_\alpha(x)$ denotes the modified Bessel functions. If we go back to the primary variables (z,z^*) according to (8.1), we obtain with substitutions

$$t \equiv \frac{s}{\sqrt{1-zz^*}}, \quad t^* \equiv \frac{s^*}{\sqrt{1-zz^*}}, \tag{9.4}$$

the following class of generating functions for the disc polynomials:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{f_{\mathrm{H}}^{(m,n)}(0,0) t^m t^{*n}}{m!n!} P_{m,n}^{\alpha}(z,z^*) = \frac{\alpha! 2^\alpha I_\alpha\left(\kappa\sqrt{tt^*(1-zz^*)}\right)}{\left(\kappa\sqrt{tt^*(1-zz^*)}\right)^\alpha} f_{\mathrm{H}}(tz, t^*z^*). \tag{9.5}$$

As a first example, we consider the function $f_{\mathrm{H}}(z,z^*) = \exp\left(\frac{1}{2}(z+z^*)\right)$ which is a solution of the Helmholtz equation

$$\left(4\frac{\partial^2}{\partial z \partial z^*} - 1\right)\exp\left(\frac{1}{2}(z+z^*)\right) = 0, \quad \Rightarrow \quad \kappa = \pm \mathrm{i}, \quad f_{\mathrm{H}}^{(m,n)}(0,0) = \frac{1}{2^{m+n}}. \tag{9.6}$$

The corresponding generating function is

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^m t^{*n}}{2^{m+n} m!n!} P_{m,n}^{\alpha}(z,z^*) = \frac{\alpha! 2^\alpha J_\alpha\left(\sqrt{tt^*}(1-zz^*)\right)}{\left(\sqrt{tt^*}(1-zz^*)\right)^\alpha} \exp\left(\frac{1}{2}(tz + t^*z^*)\right). \tag{9.7}$$

As a second example, we consider the related function $f_{\mathrm{H}}(z,z^*) = \exp\left(\frac{1}{2}(z-z^*)\right)$ which is a solution of the Helmholtz equation

$$\left(4\frac{\partial^2}{\partial z \partial z^*} + 1\right)\exp\left(\frac{1}{2}(z-z^*)\right) = 0, \quad \Rightarrow \quad \kappa = \pm 1, \quad f_{\mathrm{H}}^{(m,n)}(0,0) = \frac{(-1)^n}{2^{m+n}}. \tag{9.8}$$

The corresponding generating function is

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n t^m t^{*n}}{2^{m+n} m!n!} P_{m,n}^{\alpha}(z,z^*) = \frac{\alpha! 2^\alpha I_\alpha\left(\sqrt{tt^*}(1-zz^*)\right)}{\left(\sqrt{tt^*}(1-zz^*)\right)^\alpha} \exp\left(\frac{1}{2}(tz - t^*z^*)\right). \tag{9.9}$$

A localized solution of the two-dimensional Helmholtz equation with central symmetry is $f_{\mathrm{H}}(z,z^*)= J_0\left(k_0\sqrt{zz^*}\right)$ according to

$$\left(4\,\frac{\partial^2}{\partial z\partial z^*}+k_0^2\right)J_0\left(k_0\sqrt{zz^*}\right)=0,\quad\Rightarrow\quad \kappa=\pm k_0,\quad f_{\mathrm{H}}^{(m,n)}(0,0)=(-1)^n\left(\frac{k_0}{2}\right)^{2n}\delta_{m,n}. \tag{9.10}$$

It possesses a two-dimensional Fourier spectrum which is rotational-invariant with the constant length k_0 of the wave vectors $\boldsymbol{k}$ according to

$$\begin{aligned}\int \mathrm{d}x\wedge\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}(k_xx+k_yy)}J_0\left(k_0\sqrt{x^2+y^2}\right)&=2\pi\int_0^\infty \mathrm{d}rrJ_0(|\boldsymbol{k}|r)J_0(k_0r)\\ &=\frac{2\pi}{k_0}\,\delta(|\boldsymbol{k}|-k_0),\quad |\boldsymbol{k}|\equiv\sqrt{k_x^2+k_y^2}.\end{aligned} \tag{9.11}$$

From considered solution of the Helmholtz equation we find the following generating function for the subset of disc polynomials with equal lower indices:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(tt^*)^n}{n!^2}\left(\frac{k_0}{2}\right)^{2n}P_{n,n}^{\alpha}(z,z^*)=\frac{\alpha!2^{\alpha}I_{\alpha}\left(k_0\sqrt{tt^*}(1-zz^*)\right)}{\left(k_0\sqrt{tt^*}(1-zz^*)\right)^{\alpha}}\,J_0\left(k_0\sqrt{tt^*\,zz^*}\right). \tag{9.12}$$

The two variables (t,t^*) are independent variables and do not necessarily have to be complex conjugated.

Other than the discussed generating functions can be derived for the radial Zernike functions from known generating functions for the Jacobi polynomials [7,55].

A further problem not solved up to now is the derivation of generating functions for the product of disc polynomials with different arguments but with equal indices which embed the completeness relation (3.7). In the same way, it would be interesting to derive two-dimensional integrals over products of disc polynomials with different indices but with the same arguments which include the orthogonality relations (3.4) as a special case.

10. Conclusion

We have discussed the generalized Zernike or disc polynomials $P_{m,n}^{\alpha}(z,z^*)$ with $\alpha>-1$ as a parameter and introduced by means of them orthonormalized disc functions $p_{m,n}^{\alpha}(z,z^*)$ which are well appropriate for expansions of functions over the unit disc. A main result was that the disc functions $p_{m,n}^{\alpha}(z,z^*)$ form the basis of a realization of a unitary irreducible representation of the $\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)$ group. The corresponding Lie algebra comprises as basis operators the lowering operators $\left(K_-^{(1)},K_-^{(2)}\right)$ and raising operators $\left(K_+^{(1)},K_+^{(2)}\right)$ of the disc functions and the two-dimensional Cartan subalgebra with the operators $\left(K_0^{(1)},K_0^{(2)}\right)$ which last are diagonal in the basis of the disc polynomials. We have discussed some consequences of this group property and a limiting procedure to the Heisenberg–Weyl group related to the transition from disc polynomials to Laguerre 2D polynomials. Furthermore, in addition to the Rodrigues representation of the disc polynomials, we derived an alternative operational representations which proved to be well suited for the derivation of a class of generating functions. The degeneracy of the eigenvalues in the eigenvalue equation (4.12) suggests that it should be possible to introduce interesting superpositions of disc polynomials by means of a general 2D matrix in similar way as this was made in our introduction of general Hermite 2D and Laguerre 2D polynomials.

The paper is written in the spirit and believe that almost all interesting special functions are related to representations of certain Lie groups. We express the hope that the discussed properties of the disc polynomials suggest new applications in physics.

We mention here that for reliability we have numerically and otherwise checked by computer some of the most important and particularly complicated formulae of this paper.

Appendix A. The one-dimensional analogue of the disc polynomials

The ultraspherical polynomials or Jacobi polynomials with equal upper indices $P_n^{(\alpha,\alpha)}(x)$ or Gegenbauer polynomials $C_n^{\alpha+(1/2)}(x)$ in other standardization form the 1D analogue of the generalized Zernike or disc polynomials as 2D polynomials. Although they are well known, one can discover interesting features if one considers them under the aspect of the analogue to the disc polynomials in corresponding standardization. We make this in present appendix in a short sketched form with a minimum of necessary explanations.

The 1D analogue of the unit disc is the unit interval $-1 < x < +1$ and polar coordinates consist of the variable r in the range $0 \leqslant r = |x| < 1$ and the analogue of the circle as the boundary of a disc is x/r with the two possible values ± 1 corresponding to the splitting of integrations over the interval $-1 < x < +1$ into integration over $0 \leqslant r \leqslant 1$ and summation over the two possible values $x/r = \pm 1$.

To underline the analogy to disc polynomials, we introduce in this appendix a standardization of the ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)$ and of the Gegenbauer polynomials $C_n^{\alpha+(1/2)}(x)$ which we call "interval polynomials" and use for them the notation $P_n^{\alpha}(x)$ as follows:

$$\begin{aligned} P_n^{\alpha}(x) &\equiv \frac{n!\alpha!}{(n+\alpha)!} P_n^{(\alpha,\alpha)}(x) \\ &= \frac{n!(2\alpha)!}{(n+2\alpha)!} C_n^{\alpha+1/2}(x) = (P_n^{\alpha}(x))^* \end{aligned} \tag{A.1}$$

and, furthermore, we introduce orthonormalized "interval functions" $p_n^{\alpha}(x)$ by

$$p_n^{\alpha}(x) \equiv \frac{1}{2^{\alpha}\alpha!} \sqrt{\frac{\left(n+\frac{1}{2}+\alpha\right)(n+2\alpha)!}{n!}} (1-x^2)^{\alpha/2} P_n^{\alpha}(x), \quad -1 < \alpha < +\infty. \tag{A.2}$$

We now continue in sketched form.

Orthonormalization (analogues of (3.4) and (3.6))

$$\begin{aligned} &\int_{-1}^{+1} \mathrm{d}x (1-x^2)^{\alpha} (P_l^{\alpha}(x))^* P_n^{\alpha}(x) = \frac{2^{2\alpha}}{n+\alpha+\frac{1}{2}} \frac{n!\alpha!^2}{(n+2\alpha)!} \delta_{l,n}, \\ &\int_{-1}^{+1} \mathrm{d}x (p_l^{\alpha}(x))^* p_n^{\alpha}(x) = \delta_{l,n}, \quad \alpha > -1. \end{aligned} \tag{A.3}$$

Two explicit representations (analogues of (2.4) and (2.5))

$$P_n^{\alpha}(x) = \frac{n!(2\alpha)!}{(n+2\alpha)!(\alpha-\frac{1}{2})!} \sum_{j=0}^{[n/2]} \frac{(-1)^j \left(n+\alpha-\frac{1}{2}-j\right)!}{j!(n-2j)!} (2x)^{n-2j}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k n!\alpha!}{k!(n-2k)!(k+\alpha)!2^{2k}} (1-x^2)^k x^{n-2k}. \tag{A.4}$$

Special values (analogues of (2.9) and (2.10))

$$P_{2m}^{\alpha}(0) = \frac{(-1)^m (2m)!\alpha!}{2^{2m} m!(m+\alpha)!}, \quad P_{2m+1}^{\alpha}(0) = 0, \quad P_n^{\alpha}(\pm 1) = (\pm 1)^n. \tag{A.5}$$

Inversion of (A.4) with α as parameter (analogue of (3.10))

$$x^n = \frac{n!\left(\alpha-\frac{1}{2}\right)!}{2^n(2\alpha)!} \sum_{j=0}^{[n/2]} \frac{(n+2\alpha-2j)!\left(n+\frac{1}{2}+\alpha-2j\right)}{j!(n-2j)!\left(n+\frac{1}{2}+\alpha-j\right)!} P_{n-2j}^{\alpha}(x). \tag{A.6}$$

Rodrigues-type representation (analogue of (2.6))

$$P_n^{\alpha}(x) = \frac{(-1)^n \alpha!}{2^n (n+\alpha)!} \frac{1}{(1-x^2)^{\alpha}} \frac{\partial^n}{\partial x^n} (1-x^2)^{n+\alpha}. \tag{A.7}$$

Alternative operational representation (for simplicity in transformed variable; analogue of (8.5))

$$P_n^{\alpha}\left(\frac{u}{\sqrt{1+u^2}}\right) = \frac{1}{\left(\sqrt{1+u^2}\right)^n} \frac{\alpha! 2^{\alpha} J_{\alpha}(\partial/\partial u)}{(\partial/\partial u)^{\alpha}} u^n. \tag{A.8}$$

Limiting transition to Hermite polynomials $H_n(x)$ (analogue of (2.14))

$$H_n(x) = 2^n \lim_{|\alpha|\to\infty} \left(\sqrt{\alpha}\right)^n P_n^{\alpha}\left(\frac{x}{\sqrt{\alpha}}\right). \tag{A.9}$$

Limiting transition to orthonormalized Hermite functions $h_n(x)$ (analogue of (3.12))

$$h_n(x) \equiv \frac{1}{\pi^{1/4}} \exp\left(-\frac{x^2}{2}\right) \frac{1}{\sqrt{2^n n!}} H_n(x) = \lim_{|\alpha|\to\infty} \frac{1}{\alpha^{1/4}} p_n^{\alpha}\left(\frac{x}{\sqrt{\alpha}}\right). \tag{A.10}$$

Differential equation for interval polynomials in symmetrized form (analogue of (4.7))

$$\left\{\frac{1}{2}\left((1-x^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}(1-x^2)\right) - \alpha\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x\right) + n(n+1+2\alpha) + 1 + \alpha\right\}$$
$$\times P_n^{\alpha}(x) = 0. \tag{A.11}$$

Differential equation for interval functions as eigenvalue equation (analogue of (4.13))

$$H'^{\alpha} p_n^{\alpha}(x) = \left(n + \frac{1}{2} + \alpha\right)^2 p_n^{\alpha}(x), \tag{A.12}$$

with positive definite Hermitean operator H'^{α} defined by (analogue of (4.14))

$$H'^{\alpha} \equiv -\frac{\partial^2}{\partial x^2} + \frac{1}{4}\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x\right)^2 + \frac{\alpha^2}{1-x^2}. \tag{A.13}$$

Lowering and raising operations for interval polynomials (analogues of (6.1) and (6.2))

$$\left(nx+(1-x^2)\frac{\partial}{\partial x}\right)P_n^\alpha(x)=nP_{n-1}^\alpha(x),$$

$$\left((n+1+2\alpha)x-(1-x^2)\frac{\partial}{\partial x}\right)P_n^\alpha(x)=(n+1+2\alpha)P_{n+1}^\alpha(x). \tag{A.14}$$

Lowering and raising operations for interval functions (analogues of (6.3), (6.4) and (6.6))

$$K_-\,p_n^\alpha(x)\equiv\sqrt{\frac{n-\frac{1}{2}+\alpha}{n+\frac{1}{2}+\alpha}}\left((n+\alpha)x+(1-x^2)\frac{\partial}{\partial x}\right)p_n^\alpha(x)$$

$$=\sqrt{(n+2\alpha)\,n}\;p_{n-1}^\alpha(x),$$

$$K_+\,p_n^\alpha(x)\equiv\sqrt{\frac{n+\frac{3}{2}+\alpha}{n+\frac{1}{2}+\alpha}}\left((n+1+\alpha)x-(1-x^2)\frac{\partial}{\partial x}\right)p_n^\alpha(x)$$

$$=\sqrt{(n+1+2\alpha)(n+1)}\;p_{n+1}^\alpha(x),$$

$$K_0\,p_n^\alpha(x)\equiv\left(n+\frac{1}{2}+\alpha\right)p_n^\alpha(x)=\frac{1}{2}\,[K_-,K_+]\,p_n^\alpha(x). \tag{A.15}$$

SU(1,1) operators and Casimir operator (analogues of (6.12)–(6.14))

$$K_-\equiv(H'^\alpha)^{1/4}\left\{x\left((H'^\alpha)^{1/2}-\frac{1}{2}\right)+(1-x^2)\frac{\partial}{\partial x}\right\}(H'^\alpha)^{-1/4}=K_+^\dagger,$$

$$K_+\equiv(H'^\alpha)^{1/4}\left\{x\left((H'^\alpha)^{1/2}+\frac{1}{2}\right)-(1-x^2)\frac{\partial}{\partial x}\right\}(H'^\alpha)^{-1/4}=K_-^\dagger,$$

$$K_0\equiv(H'^\alpha)^{1/2}=K_0^\dagger,\quad C=\left(\alpha+\frac{1}{2}\right)\left(\alpha-\frac{1}{2}\right)I,\quad k=\alpha+\frac{1}{2}. \tag{A.16}$$

Operator H'^α (analogue of (7.8))

$$H'^\alpha=(K_0)^2=\frac{1}{2}\,(K_-K_++K_+K_-)+\left(\alpha^2-\frac{1}{4}\right)I. \tag{A.17}$$

Generating function [45] (analogues of (9.7), (9.9) and (9.12))

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,P_n^\alpha(x)=\frac{\alpha!2^\alpha J_\alpha\left(t\sqrt{1-x^2}\right)}{\left(t\sqrt{1-x^2}\right)^\alpha}\exp(tx). \tag{A.18}$$

The three-dimensional analogue of generalized Zernike or disc polynomials are ball polynomials which are orthogonal within the unit ball $|\boldsymbol{r}|\leqslant 1$ with weight functions $(1-|\boldsymbol{r}|^2)^\alpha$, where $\boldsymbol{r}$ denotes three-dimensional vectors. The derivation of many analogous relations in appropriate representations for ball polynomials given in present paper for disc and interval polynomials seems to be yet to make.

Appendix B. Tables of disc polynomials $P^{\alpha}_{m,n}(z, z^*)$

The following Tables 1 and 2 contain the initial members of the sequences of disc polynomials $P^{\alpha}_{m,n}(z,z^*)$ up to $m+n=7$ for the values of the parameter $\alpha=-\frac{1}{2}$, 0, $\frac{1}{2}$, 1.

One can make certain simple checks of the tables. In particular, due to special case $\varphi=0$ of (2.10) the sum of the coefficients in each polynomial has to be equal to 1.

The Zernike polynomials correspond to the case $\alpha=0$ in Table 1.

Table 1

m,n	$\alpha=-\frac{1}{2}$	$\alpha=0$
0, 0	1	1
1, 0	z	z
0, 1	z^*	z^*
2, 0	z^2	z^2
1, 1	$3zz^*-2$	$2zz^*-1$
0, 2	z^{*2}	z^{*2}
3, 0	z^3	z^3
2, 1	$5z^2z^*-4z$	$3z^2z^*-2z$
1, 2	$5zz^{*2}-4z^*$	$3zz^{*2}-2z^*$
0, 3	z^{*3}	z^{*3}
4, 0	z^4	z^4
3, 1	$7z^3z^*-6z^2$	$4z^3z^*-3z^2$
2, 2	$\frac{1}{3}(35z^2z^{*2}-40zz^*+8)$	$6z^2z^{*2}-6zz^*+1$
1, 3	$7zz^{*3}-6z^{*2}$	$4zz^{*3}-3z^{*2}$
0, 4	z^{*4}	z^{*4}
5, 0	z^5	z^5
4, 1	$9z^4z^*-8z^3$	$5z^4z^*-4z^3$
3, 2	$21z^3z^{*2}-28z^2z^*+8z$	$10z^3z^{*2}-12z^2z^*+3z$
2, 3	$21z^2z^{*3}-28zz^{*2}+8z^*$	$10z^2z^{*3}-12zz^{*2}+3z^*$
1, 4	$9zz^{*4}-8z^{*3}$	$5zz^{*4}-4z^{*3}$
0, 5	z^{*5}	z^{*5}
6, 0	z^6	z^6
5, 1	$11z^5z^*-10z^4$	$6z^5z^*-5z^4$
4, 2	$33z^4z^{*2}-48z^3z^*+16z^2$	$15z^4z^{*2}-20z^3z^*+6z^2$
3, 3	$\frac{1}{5}(231z^3z^{*3}-378z^2z^{*2}+168zz^*-16)$	$20z^3z^{*3}-30z^2z^{*2}+12zz^*-1$
2, 4	$33z^2z^{*4}-48zz^{*3}+16z^{*2}$	$15z^2z^{*4}-20zz^{*3}+6z^{*2}$
1, 5	$11zz^{*5}-10z^{*4}$	$6zz^{*5}-5z^{*4}$
0, 6	z^{*6}	z^{*6}
7, 0	z^7	z^7
6, 1	$13z^6z^*-12z^5$	$7z^6z^*-6z^5$
5, 2	$\frac{1}{3}(143z^5z^{*2}-220z^4z^*+80z^3)$	$21z^5z^{*2}-30z^4z^*+10z^3$
4, 3	$\frac{1}{5}(429z^4z^{*3}-792z^3z^{*2}+432z^2z^*-64z)$	$35z^4z^{*3}-60z^3z^{*2}+30z^2z^*-4z$
3, 4	$\frac{1}{5}(429z^3z^{*4}-792z^2z^{*3}+432zz^{*2}-64z^*)$	$35z^3z^{*4}-60z^2z^{*3}+30zz^{*2}-4z^*$
2, 5	$\frac{1}{3}(143z^2z^{*5}-220zz^{*4}+80z^{*3})$	$21z^2z^{*5}-30zz^{*4}+10z^{*3}$
1, 6	$13zz^{*6}-12z^{*5}$	$7zz^{*6}-6z^{*5}$
0, 7	z^{*7}	z^{*7}

Table 2

m, n	$\alpha = \frac{1}{2}$	$\alpha = 1$
0, 0	1	1
1, 0	z	z
0, 1	z^*	z^*
2, 0	z^2	z^2
1, 1	$\frac{1}{3}(5zz^* - 2)$	$\frac{1}{2}(3zz^* - 1)$
0, 2	z^{*2}	z^{*2}
3, 0	z^3	z^3
2, 1	$\frac{1}{3}(7z^2z^* - 4z)$	$2z^2z^* - z$
1, 2	$\frac{1}{3}(7zz^{*2} - 4z^*)$	$2zz^{*2} - z^*$
0, 3	z^{*3}	z^{*3}
4, 0	z^4	z^4
3, 1	$3z^3z^* - 2z^2$	$\frac{1}{2}(5z^3z^* - 3z^2)$
2, 2	$\frac{1}{15}(63z^2z^{*2} - 56zz^* + 8)$	$\frac{1}{3}(10z^2z^{*2} - 8zz^* + 1)$
1, 3	$3zz^{*3} - 2z^{*2}$	$\frac{1}{2}(5zz^{*3} - 3z^{*2})$
0, 4	z^{*4}	z^{*4}
5, 0	z^5	z^5
4, 1	$\frac{1}{3}(11z^4z^* - 8z^3)$	$3z^4z^* - 2z^3$
3, 2	$\frac{1}{5}(33z^3z^{*2} - 36z^2z^* + 8z)$	$5z^3z^{*2} - 5z^2z^* + z$
2, 3	$\frac{1}{5}(33z^2z^{*3} - 36zz^{*2} + 8z^*)$	$5z^2z^{*3} - 5zz^{*2} + z^*$
1, 4	$\frac{1}{3}(11zz^{*4} - 8z^{*3})$	$3zz^{*4} - 2z^{*3}$
0, 5	z^{*5}	z^{*5}
6, 0	z^6	z^6
5, 1	$\frac{1}{3}(13z^5z^* - 10z^4)$	$\frac{1}{2}(7z^5z^* - 5z^4)$
4, 2	$\frac{1}{15}(143z^4z^{*2} - 176z^3z^* + 48z^2)$	$7z^4z^{*2} - 8z^3z^* + 2z^2$
3, 3	$\frac{1}{35}(429z^3z^{*3} - 594z^2z^{*2} + 216zz^* - 16)$	$\frac{1}{4}(35z^3z^{*3} - 45z^2z^{*2} + 15zz^* - 1)$
2, 4	$\frac{1}{15}(143z^2z^{*4} - 176zz^{*3} + 48z^{*2})$	$7z^2z^{*4} - 8zz^{*3} + 2z^{*2}$
1, 5	$\frac{1}{3}(13zz^{*5} - 10z^{*4})$	$\frac{1}{2}(7zz^{*5} - 5z^{*4})$
0, 6	z^{*6}	z^{*6}
7, 0	z^7	z^7
6, 1	$5z^6z^* - 4z^5$	$4z^6z^* - 3z^5$
5, 2	$\frac{1}{3}(39z^5z^{*2} - 52z^4z^* + 16z^3)$	$\frac{1}{3}(28z^5z^{*2} - 35z^4z^* + 10z^3)$
4, 3	$\frac{1}{35}(715z^4z^{*3} - 1144z^3z^{*2} + 528z^2z^* - 64z)$	$14z^4z^{*3} - 21z^3z^{*2} + 9z^2z^* - z$
3, 4	$\frac{1}{35}(715z^3z^{*4} - 1144z^2z^{*3} + 528zz^{*2} - 64z^*)$	$14z^3z^{*4} - 21z^2z^{*3} + 9zz^{*2} - z^*$
2, 5	$\frac{1}{3}(39z^2z^{*5} - 52zz^{*4} + 16z^{*3})$	$\frac{1}{3}(28z^2z^{*5} - 35zz^{*4} + 10z^{*3})$
1, 6	$5zz^{*6} - 4z^{*5}$	$4zz^{*6} - 3z^{*5}$
0, 7	z^{*7}	z^{*7}

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